@ Oka group, group seminar, April, 18, 2025

Electromagnetic Responses of 3D Topological Insulators and Axion Electrodynamics in Condensed Matter Systems

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- ✓ This talk is based on my Bacheor Thesis at Tada group in Hiroshima University.
- ✓ My bacheor thesis written in Japanese can be found in my HP. Scan the right QR code →.

Outline

This talk consists of three parts.

Background

Electromagnetic Responses of 3D Topological Insulators

Axion Electrodynamics in Topological Materials

This talk is mainly based on two papers:
Qi, Hughes, and Zhang, *Phys. Rev. B*, (2008) and Sekine and Nomura, *J. Appl. Phys.*, (2021)

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Background

- In 1950s, Ginzburg and Landau developed the standard theory for phases and phase transitions.
→ Very Successful

Xiao-Gang Wen, *Rev. Mod. Phys.*, (2017)

- However, in 1980s, Quantum Hall Effect (QHE) was discovered as a first example of **Topological Phases**.

K. v. Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.*, (1980)

- In 2005, Kane and Mele proposed **Topological Insulators (TIs)** protected by time reversal symmetry.

Bulk → Energy Gap = like an Insulator
Edge → No Energy Gap = like a Metal

C. L. Kane and E. J. Mele, *Phys. Rev. Lett.*, (2005)

C. L. Kane and E. J. Mele, *Phys. Rev. Lett.*, (2005)

Background

- In 2006 and 2007, Fu, Kane and Mele developed a basic theory of topological insulators including the topological band theory and 3D topological Insulators.

Liang Fu and C. L. Kane, *Phys. Rev. B*, (2006)
Liang Fu and C. L. Kane, *Phys. Rev. B*, (2007)
Liang Fu, C. L. Kane, and E. J. Mele, *Phys. Rev. Lett.*, (2007)

- In 2008, Qi, Hughes and Zhang construct a topological field theory that describe topological responses of TIs.
  → In 3D TIs, effective action is equivalent to **Axion Electrodynamics**. Qi, Hughes, and Zhang, *Phys. Rev. B*, (2008)
- There are some theoretical proposal that potentially realize Axion electrodynamics in materials.

Sekine and Nomura, *J. Appl. Phys.*, (2021)

Notations

$\mu, \nu, \rho, \sigma = 0, 1, 2, 3, \cdots$: Indices for space-time

$i, j, k, l = 1, 2, 3, \cdots$: Indices for space

Natural units: $\hbar = c = e = 1$ (restored when necessary)

Minkowski metric: $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1, \cdots)$

$\alpha, \beta = 1, 2, 3, \cdots$: Band Indices

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Electromagnetic Responses of 3D TIs

1. Quantum Hall Effect and Chern-Simons gauge theories
2. Effective Action of 3D Topological Insulators
3. Topological Electromagnetic Response
    3.1. Surface Half-integer Quantum Hall Effect
    3.2. Topological Magnetoelectric Effect

QHE and Chern-Simons gauge theories

First, we consider the effecvie action of QHE

$$S_{\mathrm{CS}}^{(2+1)} = \frac{Ch_1}{4\pi} \int d^2x dt \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$Ch_1 = \frac{1}{4\pi} \int d^2k \; \epsilon^{ij} \, \mathrm{Tr}\left[f_{ij}\right]$ is the first Chern number

A_μ is the U(1) gauge field

f_{ij} is the Berry curvature

- Topological Response (= Hall current) can be calculated as

$$j^\mu = \frac{\delta S_{\mathrm{CS}}^{(2+1)}}{\delta A_\mu} = \frac{Ch_1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = \sigma_{\mathrm{H}} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$$

QHE and Chern-Simons gauge theories

Next, we consider the 4D generalized QHE

Zhang and Hu, *Science*, (2001) Bernevig et. al., *Ann. Phys.*, (2002)

- Chern-Simons gauge theory can be defined in odd-dimenstion space-time.
- (2+1)-d QHE can be generalized to (4+1)-d.
- 4D QHE is time reversal invariant.

QHE and Chern-Simons gauge theories

The effecvie action of the 4D generalized QHE:

$$S_{\mathrm{CS}}^{4+1} = \frac{Ch_2}{24\pi^2} \int d^4x dt \; \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau$$

$$Ch_2 = \frac{1}{32\pi^2} \int d^4k \; \epsilon^{ijkl} \, \mathrm{Tr}\left[f_{ij} f_{kl}\right]$$: Second Chern number

$$f_{ij}^{\alpha\beta} = \partial_{k_i} a_j^{\alpha\beta} - \partial_{k_j} a_i^{\alpha\beta} + i[a_i, a_j]^{\alpha\beta}$$: Berry curvature

$$a_i^{\alpha\beta} = -i \left\langle u_{\alpha,\boldsymbol{k}} \middle| \partial_{k_i} \middle| u_{\beta,\boldsymbol{k}} \right\rangle$$: Berry connection

- Topological Response (= Hall current) can be calculated as

$$j^\mu = \frac{\delta S_{\mathrm{CS}}^{(4+1)}}{\delta A_\mu} = \frac{Ch_2}{8\pi^2} \epsilon^{\mu\nu\rho\sigma\tau} \partial_\nu A_\rho \partial_\sigma A_\tau$$

Effective Action of 3D Topological Insulators

✓ Here, we derive the effective action of 3D TIs by dimensional reduction.

4D QHE

- (4+1)-d model
- Nontrivial topology
- Time reversal symmetry

dimensional reduction →

3D TIs

- (3+1)-d model
- Nontrivial topology
- Time reversal symmetry

Effective action
→ (4+1)-d Chern-Siomons gauge theory

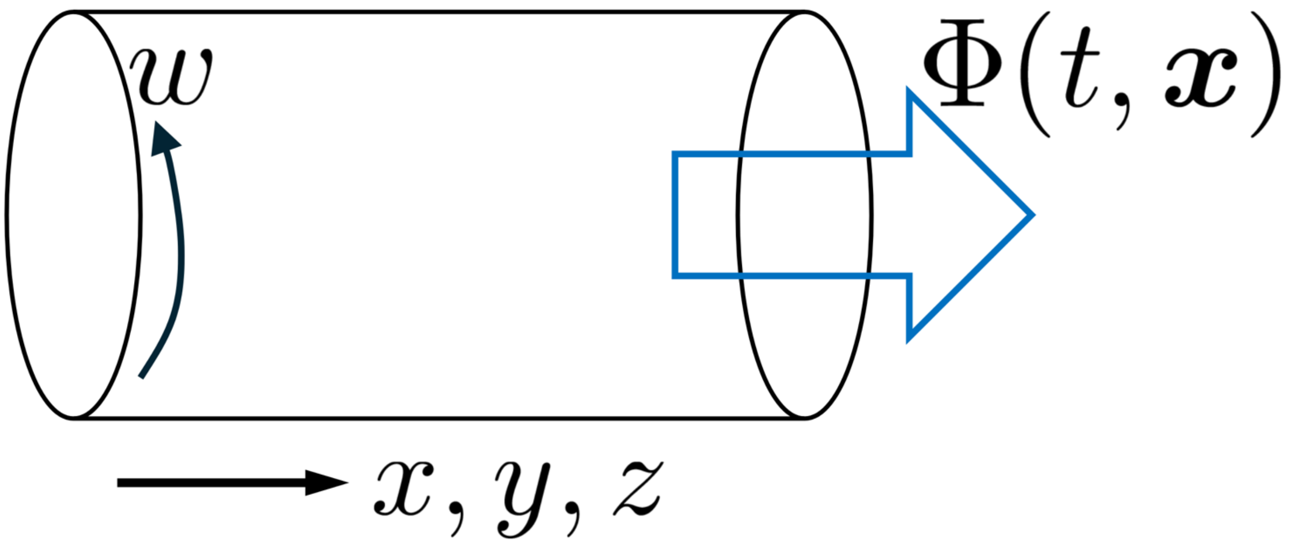
Effective action→ ???

Effective Action of 3D Topological Insulators

We consider the 4D QHE on generalized cylinder.

Gauge field configurations: $A_\mu = (A_0, A_1, A_2, A_3, \Phi(t, \boldsymbol{x})/L_w)$, which do not depend on the w direction.

Integrate action out with respect to w direction and $L_w \to 0$



Effective Action of 3D Topological Insulators

Effective action of 4D QHE can be calculated as

$$S_{\mathrm{CS}}^{4+1} = \frac{Ch_2}{24\pi^2}\int d^4x dt\ \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau$$
$$= \frac{1}{8\pi^2}\int d^3x dt\ \theta(t,\boldsymbol{x})\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma .$$

We define the theta field or Axion field as

$$\theta(t,\boldsymbol{x}) = Ch_2 \Phi(t,\boldsymbol{x}) = -\frac{1}{4\pi}\int d^3k\ \epsilon^{ijk}\,\mathrm{Tr}\left[a_i\partial_j a_k + \frac{2}{3} i a_i a_j a_k\right].$$

Effective Action of 3D Topological Insulators

Time reversal symmetry constraint quantizes Axion field.

$$\theta(t, \boldsymbol{x}) = 0, \pi \pmod{2\pi}$$

✓ $\theta = 0$ → **Trivial Insulators**.
✓ $\theta = \pi$ → **Strong Topological Insulators**.

Therefore, effective action of 3D TIs are given by

$$\begin{aligned} S_{3D} &= \frac{e^2}{8\pi^2 \hbar c} \int d^3x dt\ \theta(t, \boldsymbol{x}) \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \\ &= \frac{e^2}{4\pi \hbar c} \int d^3x dt\ \boldsymbol{E}(t, \boldsymbol{x}) \cdot \boldsymbol{B}(t, \boldsymbol{x}). \end{aligned}$$

Qi, Hughes, and Zhang, *Phys. Rev. B*, (2008)

Effective Action of 3D Topological Insulators

- Now we have two kinds of strong $\mathbb{Z}_2$ topological invariants.

$$\theta(t, \boldsymbol{x}) = 0, \pi \pmod{2\pi}$$

From Topological field theory

$$(-1)^{\nu_0} = \prod_{n_1, n_2, n_3 = 0,1} \delta(\Lambda_{i=(n_1 n_2 n_3)})$$

From Topological band theory

These strong $\mathbb{Z}_2$ topological invariants are **equivalent!!!**.

Wang, Qi, and Zhang, *New J. Phys.*, (2010)

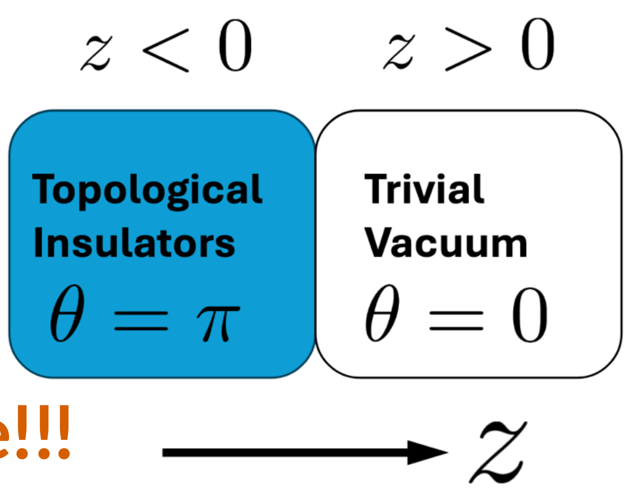
ν_0 : Can be easily computed
θ : Directly observable in response coefficients

Surface Half-integer Quantum Hall Effect

- We consider the situation where $z < 0$ is the topological insulator and $z > 0$ is the trivial vaccum.
- Axion field: $\partial_z \theta(t, \boldsymbol{x}) = \pi \delta(z)$
- Effective action at surface: $S_{\text{surface}} = \frac{1}{8\pi} \int dx dy dt \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$

Then, the current at surface $z = 0$ is

$$j^\mu = \frac{\delta S_{\text{surface}}}{\delta A_\mu} = \frac{1}{2} \frac{e^2}{h} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho$$



Half quantized Hall conductance!!!

Surface Half-integer Quantum Hall Effect

- Physically, the surface half-integer quantum Hall effect originates from a single Dirac cone at surface.

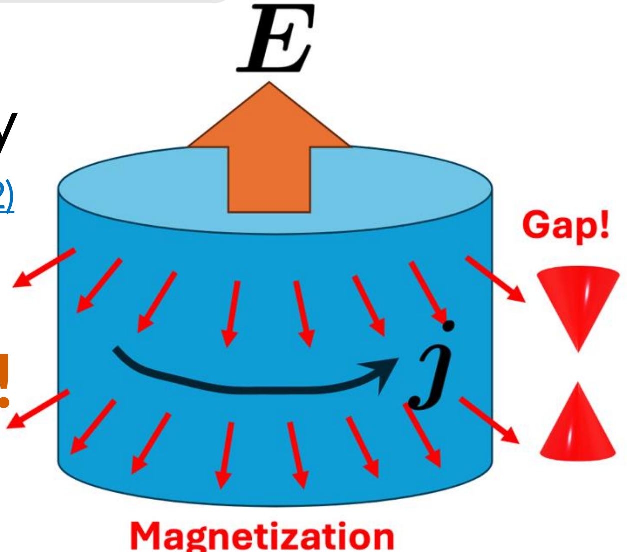
- Consider the effective Hamiltonian of 3D TIs at surface with magnetic impurities. Zhang, et al., *Nat. Phys.*, (2009)

$$H_{\mathrm{surface}}(k_x, k_y) = \hbar v_{\mathrm{F}}(k_y\sigma_x - k_x\sigma_y) + \boldsymbol{m}\cdot\boldsymbol{\sigma}$$

- Hall conductance can be computed by TKNN formula Thouless, et al., *Phys. Rev. Lett.*, (1982) Kohmoto, *Ann. Phys.*, (1985)

$$\sigma_{\mathrm{H}} = \frac{e^2}{2h}\mathrm{sgn}(m_z)$$

Half quantized!!!



Topological Magnetoelectric Effect

- Total effective actions in 3D TIs are

$$
\begin{aligned}
S_{\rm tot} &= S_{\rm Maxwell} + S_{3D} \\
&= \int d^4x \left[-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} F_{\mu\nu} \mathcal{P}^{\mu\nu} - \frac{1}{c} j^{\mu} A_{\mu} \right] \\
&\quad + \frac{e^2}{8\pi^2 \hbar c} \int d^3x dt \; \theta(t, \boldsymbol{x}) \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma
\end{aligned}
$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: Field strength tensor

$\mathcal{P}^{0i} = P_i, \mathcal{P}^{ij} = \epsilon^{ijk} M_k$: Electric polarization, Magnetization

➢ Equation of motion ➡ $\dfrac{\delta S_{\rm tot}}{\delta A_\mu} = 0$

Topological Magnetoelectric Effect

- We derive the Modified Maxwell Equations in 3D TIs.

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho$$

$$\nabla \times \boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}$$

α : Fine structure constant

$$\nabla \cdot \boldsymbol{B} = 0$$

Axion Electrodynamics!

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0$$

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} + \frac{\alpha}{\pi}\theta\boldsymbol{B} = \epsilon\boldsymbol{E} + \frac{\alpha}{\pi}\theta\boldsymbol{B}$$

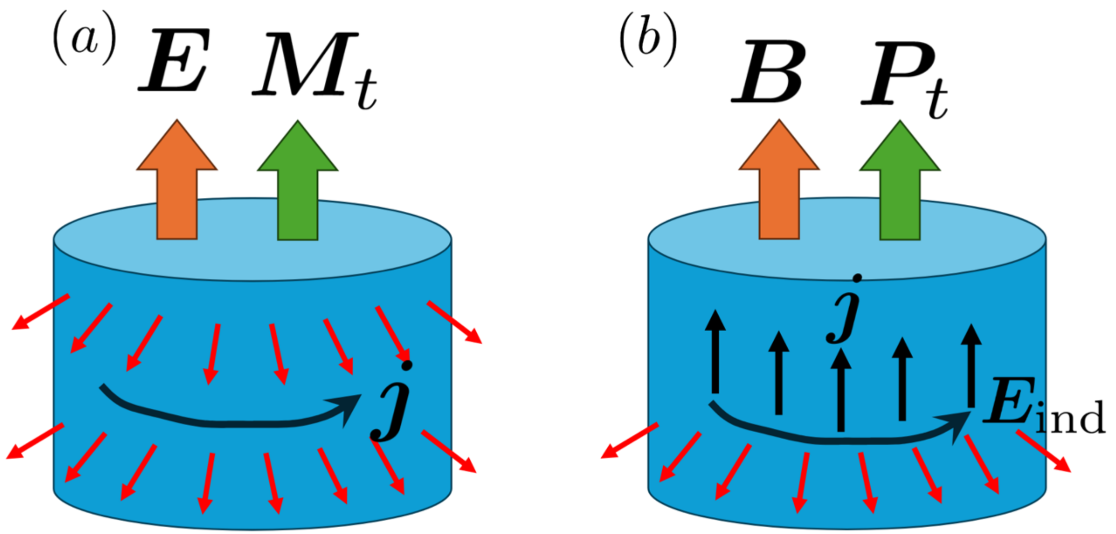
$$\boldsymbol{H} = \boldsymbol{B} - 4\pi\boldsymbol{M} + \frac{\alpha}{\pi}\theta\boldsymbol{E} = \frac{\boldsymbol{B}}{\mu} + \frac{\alpha}{\pi}\theta\boldsymbol{E}$$

Topological Magnetoelectric Effect

✓ Physically, the topological magnetoelectric effect originates from a surface half quantized hall current.

$$M_t = \frac{\alpha}{4\pi}E$$

$$P_t = \frac{\alpha}{4\pi}B$$



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Axion Electrodynamics in Topological Materials

1. Chiral Anomaly
2. Derivation of Effective Action of 3D TIs
3. Dynamical Axion Fields
4. Weyl Semimetals

Axion Electrodynamics in Topological Materials

- The action for Axion electrodynamics

$$S_\theta = \frac{e^2}{8\pi^2}\int d^3x dt\ \theta(t,\boldsymbol{x})\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma$$
$$= \frac{e^2}{32\pi^2}\int d^3x dt\ \theta(t,\boldsymbol{x})\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

- Actually, S_θ originates from the **chiral anomaly in (3+1)-d**.

⇒ First, we review the chiral anomaly.

Chiral Anomaly

- The action for Dirac field in (3+1)-d with U(1) gauge fields

$$S[\bar{\psi}, \psi, A_\mu] = \int d^4x\ \bar{\psi}[i\gamma^\mu D_\mu - m]\psi$$

$D_\mu = \partial_\mu - ieA_\mu$: Covariant derivative

- Chira transformation:

$$\psi \to \psi' = e^{i\alpha\gamma^5}\psi$$
$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{i\alpha\gamma^5}$$

- In $m = 0$, the action has chiral symmetry

$$S[\bar{\psi}, \psi, A_\mu] = S[\bar{\psi}', \psi', A_\mu]$$

Chiral Anomaly

- In quantum mechanics, we require that the partition function is invariant under the chrial transformation.

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{iS[\bar{\psi},\psi,A_\mu]} = \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi' \; e^{iS[\bar{\psi}',\psi',A_\mu]}$$

- The integral measure transforms as $\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = J(\alpha)\mathcal{D}\bar{\psi}\mathcal{D}\psi$

  $J(\alpha)$: Jacobian under the chiral transformation

- Therefore, the action transforms as

(Chiral) Anomaly!!!

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{iS[\bar{\psi},\psi,A_\mu]} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{iS[\bar{\psi},\psi,A_\mu]+\log J(\alpha)}$$

$$= iS(\alpha)$$

Chiral Anomaly

- In (3+1)-d, the chiral anomaly can be computed as

$$S(\alpha) = -i \log J(\alpha) = -\frac{e^2}{16\pi^2} \int d^3x dt \ \alpha \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} .$$

✓ This method for computing anomalies is called **Fujikawa's method**.

Fujikawa, *Phys. Rev. Lett.*, (1979)
Fujikawa, *Phys. Rev. D*, **21**, (1980)

✓ Recently, there has been significant progress in the study of anomalies, but I cannot understand all the key concepts.

Derivation of Effective Action of 3D TIs

- By computing the chiral anomaly, we derive the effective action of 3D TIs.
- The effective Hamiltonian for 3D TIs Zhang, et al., *Nat. Phys.*, (2009)

$$H(\boldsymbol{k}) = k_x\alpha^1 + k_y\alpha^2 + k_z\alpha^3 + m\alpha^4$$

Clifford algebra: $\{\alpha^a, \alpha^b\} = 2\delta^{ab}$

$m > 0 \rightarrow$ Topological Insulators
$m < 0 \rightarrow$ Trivial Insulators

- The action for 3D trivial insulators

$$S[\bar{\psi}, \psi, A_\mu] = \int d^4x\ \bar{\psi}[i\gamma^\mu D_\mu + m]\psi,\ \ \gamma^i = \alpha^0\alpha^i.$$

Derivation of Effective Action of 3D TIs

- On the other hand, the action for 3D TIs

$$S[\bar{\psi}, \psi, A_\mu] = \int d^4x\ \bar{\psi}[i\gamma^\mu D_\mu - m]\psi,\ \ \gamma^i = \alpha^0\alpha^i.$$

- These two action can be continuously connected to the chiral transformation.

$$\psi \rightarrow \psi' = e^{-i\pi\gamma^5/2}\psi$$
$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}e^{-i\pi\gamma^5/2}$$

- Therefore, the effective action from the chiral anomaly

$$S\left(-\frac{\pi}{2}\right) = S_\theta = \frac{e^2}{32\pi^2}\int d^3x dt\ \theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},\ \ \theta = \pi$$

Axion term!!!

Dynamical Axion Fields

- ✓ The Axion field of 3D TIs is quantized to π.
- ✓ If a system has either time reversal or spatial invarsion symmetry, the Axion field is quantized to 0 or π.
- Let us consider the following effective Hamiltonian

$$H(\boldsymbol{k}) = k_x\alpha^1 + k_y\alpha^2 + k_z\alpha^3 + m\alpha^4 + m'\alpha^5,\ \ \alpha^5 = \alpha^1\alpha^2\alpha^3\alpha^4.$$

$\mathsf{T}^{-1}H(\boldsymbol{k})\mathsf{T} \neq H(-\boldsymbol{k}) \rightarrow$ No time reversal symmetry

$\mathsf{P}^{-1}H(\boldsymbol{k})\mathsf{P} \neq H(-\boldsymbol{k}) \rightarrow$ No spatial inversion symmetry

Fujikawa's method →

$$\theta = \frac{\pi}{2}(1 + \mathrm{sgn}(m)) - \arctan\frac{m'}{m}$$

Takes any values!!!

Dynamical Axion Fields

- ✓ In a lattice system, Antiferromagnetic 3D TIs can realize this Axion field with space-time dependence.

Sekine and Nomura, *J. Phys. Soc. Jpn.*, (2014)
Sekine and Nomura, *Phys. Rev. Lett.*, (2016)

- ✓ Remember the modified Maxwell equations

$$\left(\boldsymbol{j} = c\nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t} \right)$$

$$\nabla \times \boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}$$

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P} + \frac{\alpha}{\pi}\theta \boldsymbol{B} = \epsilon \boldsymbol{E} + \frac{\alpha}{\pi}\theta \boldsymbol{B}$$

$$\boldsymbol{H} = \boldsymbol{B} - 4\pi \boldsymbol{M} + \frac{\alpha}{\pi}\theta \boldsymbol{E} = \frac{\boldsymbol{B}}{\mu} + \frac{\alpha}{\pi}\theta \boldsymbol{E}$$

- ✓ $\boldsymbol{P}$ and $\boldsymbol{M}$ can be interpreted as an electric current.

Dynamical Axion Fields

- An electric current originates from the Axion field

$$\boldsymbol{j} = \boldsymbol{j}_{\mathrm{AHE}} + \boldsymbol{j}_{\mathrm{CME}} = \frac{e^2}{2\pi h}\left[\nabla\theta(t,\boldsymbol{x}) \times \boldsymbol{E} + \frac{1}{c}\frac{\partial\theta(t,\boldsymbol{x})}{\partial t}\boldsymbol{B}\right]$$

$$\boldsymbol{j}_{\mathrm{AHE}} = \frac{e^2}{2\pi h}\nabla\theta(t,\boldsymbol{x}) \times \boldsymbol{E}$$: **Anomolus Hall Effect**

$$\boldsymbol{j}_{\mathrm{CME}} = \frac{e^2}{2\pi h}\frac{1}{c}\frac{\partial\theta(t,\boldsymbol{x})}{\partial t}\boldsymbol{B}$$: **Chiral Magnetic Effect**

- Surface half quantized hall currents correspond to $\boldsymbol{j}_{\mathrm{AHE}}$.

Weyl Semimetals

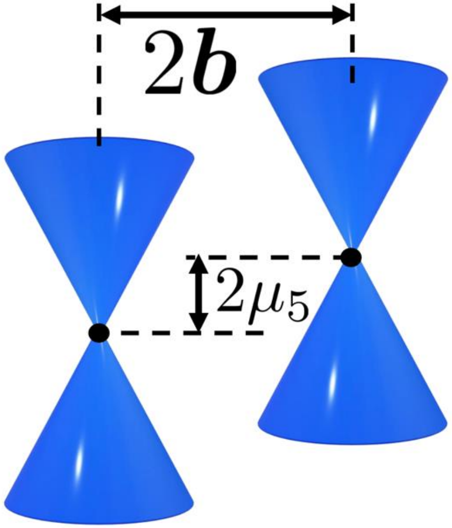
- Weyl semimetals as an example of dynamical Axion fields

$$H(\boldsymbol{k}) = \boldsymbol{k} \cdot (\tau^3 \otimes \boldsymbol{\sigma}) + \boldsymbol{b} \cdot (I \otimes \boldsymbol{\sigma}) - \mu_5 (\tau^3 \otimes I)$$

- The action for Weyl semimetals

$$S = \int d^4x \ \bar{\psi} i \gamma^\mu [\partial_\mu - ieA_\mu - ib_\mu \gamma^5] \psi$$

$b_\mu = (\mu_5, -\boldsymbol{b})$: Chiral gauge field



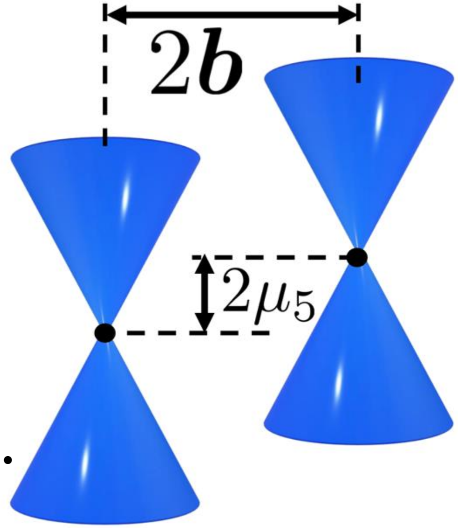
Weyl Semimetals

- By applying Fujikawa's method, Zyuzin and Burkov, *Phys. Rev. B*, (2012) Vazifeh and Franz, *Phys. Rev. Lett.*, (2013)

$$S_\theta = \frac{e^2}{32\pi^2} \int d^3x dt\ \theta(t, \boldsymbol{x}) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},\ \theta(t, \boldsymbol{x}) = -2b_\mu x^\mu$$

- Computing the electric current, we obtain

$$\boldsymbol{j}_{\mathrm{AHE}} = \frac{e^2}{\pi h} \boldsymbol{b} \times \boldsymbol{E},\ \boldsymbol{j}_{\mathrm{CME}} = -\frac{e^2}{\pi h} \mu_5 \boldsymbol{B}$$



- In equilibrium states,
the chiral magnetic effect cannot be realized.

Vazifeh and Franz, *Phys. Rev. Lett.*, (2013)

Conclusion

- The topological field theory that describes topological responses in 3D TIs can be obtained via dimensional reduction from the (4+1)-d Chern-Simons gauge theory.
- Topological electromagnetic responses in 3D TIs originate from a single Dirac cone at the surface.
- There are some topological materials where Axion emerge.